Exploring and Creating Tessellations

This two-part activity has students explore and construct several different types of tessellations. Part one of the activity uses only regular polygons to create pure and semi-pure tessellations; therefore, you will need scripts for the constructions of an equilateral triangle, a square, and a regular hexagon. Part two extends the notion of tessellations to create non-regular polygon and “Escher-like” tessellations.

Mathematics: Students will use translations, rotations, and reflections to create tessellations. Moreover, the students will investigate the following theorems:

- Equilateral triangles, squares, and hexagons are the only three regular polygons that create pure tessellations.
- There are eight different semi-regular tessellations.
- Any scalene triangle or quadrilateral will create a pure tessellation.

Mathematical Thinking: In this activity, students will be asked to conjecture about the construction of possible tessellations.

Technology: This activity uses several Sketchpad commands and features such as: Script, Translate, Rotate, Reflect, Mark Center, Mark Mirror, Mark Vector, and Mark Angle.

The following sketches illustrate the three regular polygons that tessellate the plane, and three of the eight semi-pure regular tessellations described in the activity.
Part 1: Constructing Pure and Semi-Pure Regular Tessellations

- What is a tessellation? Give examples of where can you find tessellations in the real world.

Definitions: A tessellation (or tiling) is an arrangement of closed shapes that completely cover the plane without overlapping and without leaving gaps. When a tessellation uses only one shape, it’s called a pure tessellation.

- Conjecture which regular polygons can be used to create a pure tessellation. How can you be sure that a regular polygon tessellates?

- Quickly sketch a pure tessellation using a regular polygon on your paper. Shade in one of the tessellated polygons, and focus your attention on one of the polygon’s vertices. How many polygons share that vertex? Comment on the sum of the measures of all of the angles which share that vertex. How is this sum connected to the ability to tessellate your polygon?

- What is the measure of each interior angle of a regular pentagon? Can three or more non-overlapping pentagons share a common vertex? Complete a chart, similar to the one below, to organize your thoughts.

<table>
<thead>
<tr>
<th>Name of Polygon</th>
<th># of sides of a regular polygon</th>
<th># of degrees in each interior angle of the regular polygon</th>
<th>if tessellated, # of polygons that share a common vertex</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>8</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>12</td>
<td></td>
</tr>
</tbody>
</table>

- How is the chart above related to the ability to tessellate a regular polygon? What are the necessary and sufficient conditions to create a pure regular tessellation?

- How many pure tessellations can be created using regular polygons? Justify your answer to your partner.

- What kind of transformations can you use to create a tessellation? Describe each in detail.
- Using commands under the **Transform** menu, The Geometer’s Sketchpad enables you to **Translate**, **Rotate**, and **Reflect** figures. Practice using each of these commands as you create as many regular pure tessellations as possible.

  a) Using your script, construct an equilateral triangle. Tile your sketch window using equilateral triangles. Compare your tiling process with your neighbor’s.
  
  b) Using your script, construct a square. Tile your sketch window using squares. Compare your tiling process with your neighbor’s.
  
  c) Using your script, construct a regular hexagon. Tile your sketch window using regular hexagons. Compare your tiling process with your neighbor’s.

When using the following commands, remember that the *units* for the angle measurement and distance can be set by selecting the **Preferences** command under the **Display** menu.

**To translate a polygon by a rectangular vector:**
- Select the polygon interior, sides and vertices of the figure you wish to translate.
- Choose **Translate** from the **Transform** menu.
- At the bottom left-hand side of the window, mark by **Rectangular Vector**.
- Enter in values for the horizontal and vertical translation, and click **OK**.

**To translate a polygon by a polar vector:**
- Select the polygon interior, sides and vertices of the figure you wish to translate.
- Choose **Translate** from the **Transform** menu.
- At the bottom left-hand side of the window, mark by **Polar Vector**.
- Enter in values for the angle and distance, and click **OK**.

**To translate a polygon by a marked vector:**
- Mark a vector of the desired translation. To do this, select two points and under the **Transform** menu, select **Mark Vector “A→B”**. Your order in which you select your points will determine the direction of the vector.
- Select the polygon interior, sides and vertices of the figure you wish to translate.
- Choose **Translate** from the **Transform** menu.
- In the window verify the desired translation, and click **OK**.

**To rotate a polygon by a fixed angle:**
- Select the point you want as a center pivot point.
- Under the **Transform** menu, select **Mark Center “X”**.
- Select the polygon interior, sides and vertices of the figure you wish to rotate.
- Choose **Rotate** from the **Transform** menu.
To rotate a polygon by a marked angle:
• Select the point you want as a center pivot point.
• Under the Transform menu, select Mark Center “X”.
• Mark an angle for the desired rotation. To do this, select three points (in the conventional order point, vertex, point), and under the Transform menu, select Mark Angle “X-Y-Z”. Your order in which you select your points will determine the direction of the angle.
• Select the polygon interior, sides and vertices of the figure you wish to rotate.
• Choose Rotate from the Transform menu.
• In the window verify the desired rotation, and click OK.

To reflect a polygon:
• Select any straight object in your sketch window as the desired mirror.
• Under the Transform menu, select Mark Mirror “q”.
• Select the polygon interior, sides and vertices of the figure you wish to reflect.
• Choose Reflect from the Transform menu.

• Regular tessellations that involve more than one shape are called semi-pure regular tessellations. How might an equilateral triangle, a square, and a regular hexagon together tessellate the plane? Demonstrate using the Sketchpad. (Hint: Place squares on each side of a regular hexagon.) There are a finite number of semi-pure regular tessellations (actually less than ten!). Working together as a class, try creating as many semi-pure regular tessellations as possible.

Part 2: Constructing Non-regular Tessellations

• Up to this point, you have made tessellations with regular polygons. You have demonstrated that there are only three pure tessellations of regular polygons and eight semi-pure regular tessellations. Do you know of a non-regular figure that tessellates? Will a scalene triangle tessellate? You might want to attack this problem concretely first (i.e., draw and cut out some congruent scalene triangles and try to construct a tessellation physically). Then using Geometer’s Sketchpad, try to construct a tessellation in a new sketch window.

• Observe the angles about each of the triangle’s vertices. How many times did each angle of the triangle fit about each vertex? What is the sum of the measures of the three angles of a triangle? Compare you results with the results of your neighbor. State a conjecture about your findings.
• Record any difficulties you had while trying to tessellate the plane with a scalene triangle using the Sketchpad. How could you remedy these?

• Do a similar investigation as above, and create a tessellation with a non-regular convex quadrilateral. Does any convex quadrilateral tessellate the plane? State a conjecture about your findings.

• Investigate whether a concave quadrilateral will tile the plane. Create your own concave quadrilateral and try to create a tessellation with it. Document your attempt by describing your process on paper.

• We have explored some of the beauty of tessellations. M. C. Escher, an artist and mathematician, has become famous for his wonderful tessellations! Escher spent many years learning how to use translations, rotations, and reflections to create his masterpieces. Study the Escher tessellations below.

• In the left-hand tessellation, focus your attention on one of the tail’s vertices. How many tadpoles share that vertex? Comment on the sum of the measures of all of the angles which share that vertex. How is this sum connected to the ability to tessellate the tadpole?

• Explain Escher’s use of symmetry in each of the above tessellations. What different transformations were used to create these tessellations? You can view additional works of Escher on the web (see the Additional Resources below).
• Use “A Tessellation Tutorial” (http://forum.swarthmore.edu/sum95/suzanne/tess.intro.html) from The Math Forum to create your own Escher-like tessellations using the Sketchpad. Write a paragraph about the construction of your tessellation.

Additional Resources:

• “A Tessellation Tutorial” from The Math Forum at Swarthmore College. This site includes tutorials and templates for making your own tessellations with various pieces of software.
  http://forum.swarthmore.edu/sum95/suzanne/tess.intro.html

• “A Tour: M.C. Escher -- Life and Work” website from the National Gallery of Art. It provides a brief biography of M.C. Escher and a virtual museum tour of some of his wonderful art work.
  http://www.nga.gov/collection/gallery/ggescher/ggescher-main1.html

• “Totally Tessellated” is a quite impressive World Wide Web site intended to be a helpful and interesting resource regarding tessellations. It is only an basic introduction into the complex world of tessellations and tilings. The three designers and creators of this site are Alok Bhushan of McLean, Virginia; Kendrick Kay of Martinez, GA; and Eleanor Williams of Palos Verdes, California. All three graduated from high school in 1998, and all three coincidentally attended Harvard University as freshman in Fall 1998.
  http://library.advanced.org/16661/

• “Celebrate 100 years of M. C. Escher” is a web site developed by iproject ONLINE. It includes links to: a written biography, Escher exhibitions, books on Escher, school resources, and Escher-related products.
  http://www.iproject.com/escher/escher100.html

• “M. C. Escher: Artist or Mathematician?” is a web site which includes a brief biography; a display of Escher’s art; and a tessellation tutorial.
  http://library.advanced.org/11750/

• This web site called “M. C. Escher on the WWW” has various links to other sites about the
  http://www.kultur-online.com/greatest/fr-escher.htm

• This web site includes a brief biography of Escher; selections from favorite Escher works; links to other WWW Escher sites; and books about M.C. Escher.
  http://www.erols.com/ziring/escher.htm

• “David McAllister's Escher Collection” has many of Escher’s drawings and paintings to view.
  http://www.cs.unc.edu/~davemc/Pic/Escher/
• “Escher Interactive Software” home page is an advertisement for software that helps you create your own Escher tessellations.
  http://www.thameshudson.co.uk/home2.htm

Related Articles: